NIKHEF-99-024

Massive gauge bosons from the conservation of topological winding numbers

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Abstract

We consider a $U(1) \times SU(2)$ gauge theory on the four-dimensional manifold $S^1 \times S^3$. If we make the assumption that only gauge transformations connected to the identity are allowed, the winding numbers of $U(1)$ around S^1 and of $SU(2)$ around S^3 become topological conserved quantities. We derive the effective theory for non-trivial winding numbers if all distances are small compared to the radii of the spheres. In the non-abelian case the gauge bosons become massive.

PACS : 11.15.Ex, 14.70.-e

Keywords : massive gauge bosons, winding numbers

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1 Introduction

The connection between topology and masses has already a long history, starting with the work of Schwinger [1], who showed that in two-dimensional QED the photon acquires a mass through the polarization diagrams of (massless) fermion loops. In three dimensions massive gauge bosons can be obtained by adding a Chern-Simons term to the action [2]. The discovery of instanton solutions [3] - [8] revealed a non-trivial vacuum in four-dimensional non-abelian gauge theories, and explained the mass of the η-meson.

In this paper we consider a $U(1) \times SU(2)$ gauge theory on the four-dimensional manifold $S^1 \times S^3$. For the metric we assume Euclidean signature. We will further assume that we can savely neglect instanton effects. It is obvious that this manifold lacks $SO(4)$–Lorentz symmetry, since the coordinate along the circle S^1 is singled out. However it is an interesting toy model for a $U(1) \times SU(2)$ gauge theory: S^1 allows maps with non-trivial winding numbers into $U(1)$, as does S^3 into $SU(2)$. In the context of gauge theories these maps correspond to pure gauge configurations, and two configurations with different winding numbers are considered to be equivalent, since they can be obtained from each other by a gauge transformation. Of course this gauge transformation has to change the winding number.

On the other hand it is an interesting question to study the case in which we restrict ourselves to gauge transformation which do not change the winding numbers. If we make the assumption that only gauge transformations connected to the identity are allowed, the winding numbers of $U(1)$ around S^1 and of $SU(2)$ around S^3 become topological conserved quantities.

In this paper we would like to make three assumption: First, as already mentioned, we assume that we can neglect instanton effects. Our second assumption is that only gauge transformations connected to the identity are allowed.
Requiring a non-trivial winding number, for example, of $SU(2)$ around S^3 does not tell us anything about the variation of the background field along the direction of S^1. Our third assumption is therefore that the $SU(2)$-background field varies along S^1 randomly enough, such that we may replace it by an average over all $SU(2)$-rotated configurations, which have the required winding number around S^3.

We derive the effective theory for non-trivial winding numbers if all distances are small compared to the radii of the spheres, but large compared to the characterisitic length required by assumption 3. We find that the $U(1)$-photon stays massless and that the $SU(2)$-gauge bosons develop massive modes. We would like to point out that no additional scalar fields are involved. Allthough in this paper we restrict ourselves to a pure gauge theory (e.g. we do not include fermions and we make no attempt to explain a non-zero value for the Weinberg angle), we would like to mention that J.A. Bagger, A.F. Falk and M. Swartz [9] have argued recently that the present electroweak precision measurements are consistent with theories in which there are no new particles below 3TeV.

Mathematically we are considering secondary characteristic classes [10].

This paper is organized as follows: In order to keep the discussion as simple as possible we first introduce in the next section a toy model, which relies only on the first two assumptions

and for which the $SU(2)$-background field stays constant along S^1. In section 3 we derive in a general way the effective action in the presence of a background field. In section 4 we apply these results to our toy model. In section 5 we relax the unneccessary restriction, which is inherent in our specific toy model, namely that the $SU(2)$-background field stays constant along S^1, and replace it by our assumption 3. We obtain a Lorentz-invariant theory with massive $SU(2)$-gauge bosons. The conclusions are given in section 6. The appendix contains some formulae for self-dual and anti-self-dual tensors.

2 The toy model

We embed $S^1 \times S^3$ in $\mathbb{R}^2 \times \mathbb{R}^4$ as follows

$$\begin{aligned} y_0 = r\cos\phi_0, &\qquad y_1 = r\sin\phi_0, \\ y_2 = r\cos\theta_2, &\qquad y_3 = r\sin\theta_2\cos\theta_3, \\ y_4 = r\sin\theta_2\sin\theta_3\cos\phi_1, &\qquad y_5 = r\sin\theta_2\sin\theta_3\sin\phi_1. \end{aligned} \tag{1}$$

ϕ_0 is the spherical coordinate for S^1, ϕ_1, θ_2 and θ_3 are the spherical coordinates for S^3. ϕ_0 and ϕ_1 take values in $[0, 2\pi]$, whereas θ_2 and θ_3 take values in $[0, \pi]$. For simplicity we have assumed that the radii of the spheres are equal. The metric tensor on $S^1 \times S^3$ is given by

$$g \;=\; r^2 d\phi_0^2 + r^2\sin^2\theta_2\sin^2\theta_3 d\phi_1^2 + r^2 d\theta_2^2 + r^2\sin^2\theta_2 d\theta_3^2. \tag{2}$$

In the neighbourhood of $\phi_0 = 0, \phi_1 = 0, \theta_2 = \pi/2, \theta_3 = \pi/2$ we introduce a local coordinate system $X' = (x'_0, x'_1, x'_2, x'_3)$ by

$$\begin{aligned} x'_0 = r\tan\phi_0, &\qquad x'_1 = r\tan\phi_1, \\ x'_2 = r\tan\left(\theta_2 - \frac{\pi}{2}\right), &\qquad x'_3 = r\tan\left(\theta_3 - \frac{\pi}{2}\right). \end{aligned} \tag{3}$$

Finally we relate the coordinate system $X' = (x'_0, x'_1, x'_2, x'_3)$ to a coordinate system $X = (x_0, x_1, x_2, x_3)$ through a rotation:

$$x'_\mu \;=\; \Lambda_{\mu\nu} x_\nu, \tag{4}$$

where $\Lambda_{\mu\nu} \in SO(4)$. It is obvious that the coordinate systems X and X' do not cover the entire manifold $S^1 \times S^3$ but only some chart U.
Maps with non-trivial winding number from S^1 to $U(1)$ and from S^3 to $SU(2)$ are given by

$$\begin{aligned} \tilde{B} \;:\; & S^1 \to U(1), \\ & \phi_0 \to \exp\left(in\phi_0\right), \end{aligned} \tag{5}$$

and by

$$\begin{aligned} B \;:\; & S^3 \to SU(2), \\ & (\phi_1, \theta_2, \theta_3) \to \frac{1}{r^m}\left(y_2 1 + iy_3\sigma_1 + iy_4\sigma_2 + iy_5\sigma_3\right)^m, \end{aligned} \tag{6}$$

where y_2, ..., y_5 are given in eq. (1), n and m are the (integer) winding numbers and the σ_a are the Pauli matrices. We calculate $\tilde{B}^{-1}\partial_\mu\tilde{B}$ and $B^{-1}\partial_\mu B$ in the coordinate system X:

$$\begin{aligned} \tilde{B}_\mu = \tilde{B}^{-1}\partial_\mu\tilde{B} &= i\frac{n}{r}\Lambda_{0\mu} + O\left(\frac{1}{r^3}\right) \\ B_\mu = B^{-1}\partial_\mu B &= i\frac{m}{r}\left(\sigma_1\Lambda_{1\mu} + \sigma_2\Lambda_{2\mu} + \sigma_3\Lambda_{3\mu}\right) + O\left(\frac{1}{r^3}\right) \end{aligned} \tag{7}$$

We now consider a pure Yang-Mills theory on $S^1 \times S^3$. The action is given by

$$\hat{S} = \int\limits_{S^1 \times S^3} d^4x \sqrt{|g|} \hat{\mathcal{L}}_{YM}, \qquad \hat{\mathcal{L}}_{YM} = \frac{1}{2g^2} \mathrm{Tr}\, \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} \tag{8}$$

with

$$\begin{aligned} \hat{F}_{\mu\nu} &= \partial_\mu \hat{A}_\nu - \partial_\nu \hat{A}_\mu + \left[\hat{A}_\mu, \hat{A}_\nu\right], \\ \hat{A}_\mu &= T^a \hat{A}^a_\mu. \end{aligned} \tag{9}$$

The T^a are the hermitian generators of the gauge group. We take the normalization to be

$$\mathrm{Tr}\, T^a T^b = \frac{1}{2} \delta^{ab}. \tag{10}$$

For $SU(2)$ we have therefore $T^a = \frac{1}{2}\sigma^a$ and

$$\left[T^a, T^b\right] = i f^{abc} T^c, \qquad f^{abc} = \frac{1}{2} \varepsilon_{abc}. \tag{11}$$

The explicit expressions for the $U(1)$- and the $SU(2)$-gauge potentials are

$$\tilde{B}_\mu = \frac{2in}{r} \Lambda_{0\mu}, \qquad B^a_\mu = \frac{2im}{r} \Lambda_{a\mu}, \tag{12}$$

up to terms of order $1/r^3$. We split the gauge field $\hat{A}_\mu$ into a fluctuating field A_μ and a background field B_μ:

$$\hat{A}_\mu = A_\mu + B_\mu \tag{13}$$

The latter one is given explicitly by eq. (12). Since the background field is pure gauge ($B_\mu = B^{-1} \partial_\mu B$) it has vanishing curvature:

$$\partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] = 0 \tag{14}$$

We will further assume that the fluctuating field A_μ vanishes outside a region V which is entirely contained in the local chart U. In that case we may replace the integration over $S^1 \times S^3$ by the integration over V. We also assume that the size of V is small compared to r, which allows us to expand everything in $1/r$. For the metric tensor we obtain

$$\sqrt{|g|} = 1 + O\left(\frac{1}{r^2}\right) \tag{15}$$

and we consider therefore the action

$$S = \frac{1}{2g^2} \int\limits_V d^4x \mathrm{Tr}\, \hat{F}_{\mu\nu} \hat{F}^{\mu\nu}. \tag{16}$$

(For a treatment of gauge theories on curved manifolds without the above approximation we refer to [11].)

3 The action in a background field

In this paragraph we express the Lagrangian

$$\mathcal{L}_{YM} \;=\; \frac{1}{2g^2}\mathrm{Tr}\; F_{\mu\nu}(A+B)F_{\mu\nu}(A+B), \tag{17}$$

corresponding to a gauge field configuration $A_\mu+B_\mu$ in terms of a Lagrangian for the fluctuating field A_μ alone. For the field strength $F_{\mu\nu}(A+B)$ we write

$$F_{\mu\nu}(A+B) = F_{\mu\nu}(A) + K_{\mu\nu}, \tag{18}$$

where

$$K_{\mu\nu} \;=\; [A_\mu, B_\nu] - [A_\nu, B_\mu]\,. \tag{19}$$

We assumed that B_μ is a pure gauge field and therefore $F_{\mu\nu}(B)=0$. Substitution of eq. (18) into eq. (17) gives us

$$\mathcal{L}_{YM} \;=\; \frac{1}{2g^2}\mathrm{Tr}\; F_{\mu\nu}(A)F_{\mu\nu}(A) + 2K_{\mu\nu}F_{\mu\nu}(A) + K_{\mu\nu}K_{\mu\nu}, \tag{20}$$

We further assume that the instantons numbers of the configurations A_μ and $A_\mu+B_\mu$ are the same:

$$Q(A+B) \;=\; Q(A), \tag{21}$$

where

$$Q(A) \;=\; \frac{1}{32\pi^2}\varepsilon_{\mu\nu\rho\sigma}\int d^4x \mathrm{Tr}\; F_{\mu\nu}(A)F_{\rho\sigma}(A). \tag{22}$$

We are primarily interested in configuration for which $Q(A+B)=Q(A)=0$. Clearly, for these configurations eq. (21) is satisfied. From eq. (21) we obtain

$$-\frac{1}{4g^2}\varepsilon_{\mu\nu\rho\sigma}\int d^4x\; \mathrm{Tr}\; 2K_{\mu\nu}F_{\rho\sigma}(A) + K_{\mu\nu}K_{\rho\sigma} \;=\; 0. \tag{23}$$

We add the l.h.s. of eq. (23) to eq. (20) and obtain

$$\int d^4x\; \mathcal{L}_{YM} \;=\; \frac{1}{2g^2}\int d^4x\; \mathrm{Tr}\; F_{\mu\nu}(A)F_{\mu\nu}(A) + 4K^-_{\mu\nu}F_{\mu\nu}(A) + 2K^-_{\mu\nu}K^-_{\mu\nu}, \tag{24}$$

where $K^-_{\mu\nu}$ is the anti-self-dual part of $K_{\mu\nu}$:

$$K^-_{\mu\nu} \;=\; \frac{1}{2}\left(K_{\mu\nu} - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}K_{\rho\sigma}\right) \tag{25}$$

Since all additional terms involve commutators, the abelian case of a $U(1)$ gauge potential is trivial. We restrict ourselves therefore to the $SU(2)$ gauge potential. The presence of the background field modifies the terms bilinear and trilinear in the fluctuating field A_μ. The Lagrangian eq.(24) is invariant under the combined transformation

$$\begin{aligned} A'_\mu &= U^{-1}A_\mu U + U^{-1}\partial_\mu U, \\ B'_\mu &= U^{-1}B_\mu U. \end{aligned} \tag{26}$$

For the gauge transformations U we restrict ourselves to transformations which are connected to the identity $U = 1$. In order to fix the gauge we choose the background field gauge [12, 13] and add a gauge fixing term

$$\mathcal{L}_{GF} = \frac{1}{2g^2} \text{Tr } 2 \left(\partial_\mu A_\mu + [B_\mu, A_\mu] \right) \left(\partial_\nu A_\nu + [B_\nu, A_\nu] \right) \tag{27}$$

to the Lagrangian. Putting everything together we obtain for the terms bilinear in A_μ:

$$\int d^4x \, \left(\mathcal{L}_{YM}|_{bilinear} + \mathcal{L}_{GF} \right) = \frac{1}{2g^2} \int d^4x \, A_\mu^a \left(S_{\mu\nu}^{ab} + T_{\mu\nu}^{ab} \right) A_\nu^b, \tag{28}$$

where

$$\begin{aligned} S_{\mu\nu}^{ab} &= \left(-\delta^{ab} \Box + 2i f^{abc} B_\rho^c \partial_\rho - f^{eac} f^{ebd} B_\rho^c B_\rho^d \right) \delta_{\mu\nu}, \\ T_{\mu\nu}^{ab} &= 2i f^{abc} \left(B_\mu^c \partial_\nu - B_\nu^c \partial_\mu + \varepsilon_{\mu\nu\rho\sigma} B_\rho^c \partial_\sigma \right) \\ &\quad - f^{eac} f^{ebd} \left(B_\mu^c B_\nu^d - B_\mu^d B_\nu^c + \varepsilon_{\mu\nu\rho\sigma} B_\rho^c B_\sigma^d \right). \end{aligned} \tag{29}$$

$S_{\mu\nu}^{ab}$ is symmetric in (μ, ν), whereas $T_{\mu\nu}^{ab}$ is self-dual. Since we used eq. (21), no anti-self-dual term appears.

4 Phenomenology of the toy model

We now come back to our toy model and use the explicit expressions given by the equations (11) and (12) for the structure constants and the background field. We consider the case in which the rotation matrix $\Lambda_{\mu\nu}$ is trivial:

$$\Lambda_{\mu\nu} = \delta_{\mu\nu} \tag{30}$$

In that case the explicit expression for the background field reads

$$B_\mu^a = \frac{2im}{r} \delta_{a\mu} \tag{31}$$

In the high-energy limit $k^2 >> m^2/r^2$, where k is the momentum of the gauge boson, we may neglect all terms involving background fields and our toy model reduces to the standard Yang-Mills theory with massless gauge bosons. In the low-energy limit we may neglect the partial derivatives and only the mass term survives in the quadratic part of the Lagrangian. To be precise this limit is valid (with all the approximations we made) in the region

$$\frac{1}{r^2} << k^2 << \frac{m^2}{r^2}. \tag{32}$$

In other words we are probing distances, which are small compared to r, but large compared to r/m. We write the mass term as

$$\mathcal{L}_{mass} = \frac{m^2}{2g^2 r^2} A_\mu^a M_{\mu\nu}^{ab} A_\nu^b \tag{33}$$

where the matrix $M_{\mu\nu}^{ab}$ is given by

$$M_{\mu\nu}^{ab} = \varepsilon_{eac} \varepsilon_{ebd} \left(\delta_{\mu\nu} \delta_{\rho\sigma} + \delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho} + \varepsilon_{\mu\nu\rho\sigma} \right) \delta_{c\rho} \delta_{d\sigma} \tag{34}$$

The mass matrix can be diagonalized by changing the variables according to

$$\begin{pmatrix} W_\mu^1 \\ W_\mu^2 \\ W_\mu^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}}\eta_{1\mu\nu} & -\frac{1}{\sqrt{2}}\eta_{2\mu\nu} & 0 \\ \frac{1}{\sqrt{6}}\eta_{1\mu\nu} & \frac{1}{\sqrt{6}}\eta_{2\mu\nu} & -\frac{2}{\sqrt{6}}\eta_{3\mu\nu} \\ \frac{1}{\sqrt{3}}\eta_{1\mu\nu} & \frac{1}{\sqrt{3}}\eta_{2\mu\nu} & \frac{1}{\sqrt{3}}\eta_{3\mu\nu} \end{pmatrix} \begin{pmatrix} A_\nu^1 \\ A_\nu^2 \\ A_\nu^3 \end{pmatrix}, \tag{35}$$

where the 't Hooft symbols [6] $\eta_{a\mu\nu}$ are defined in the appendix. We obtain for the mass term

$$\mathcal{L}_{mass} = \frac{m^2}{2g^2r^2}\left(W_\mu^1 W_\mu^1 + W_\mu^2 W_\mu^2 + 4\, W_\mu^3 W_\mu^3\right). \tag{36}$$

We see that W^1 and W^2 are degenerate in mass and that the ratio of the masses of W^1 and W^3 is independent of the winding number and is given by

$$\frac{m_{W^1}^2}{m_{W^3}^2} = \frac{1}{4}. \tag{37}$$

At this point a comment about Lorentz symmetry is in order: Altough neither our original manifold $S^1 \times S^3$ nor the specific choice of the $SU(2)$-background field is invariant under $SO(4)$-rotations, the effective Lagrangian of the high-energy limit as well as the effective Lagrangian eq. (36) of the low-energy limit posseses a $SO(4)$-symmetry. However this will be no longer true in the intermediate range

$$k^2 \approx \frac{m^2}{r^2}, \tag{38}$$

where we probe distances approximately equal to r/m. In that case the complete expression eq.(28) has to be used.

Summary and critics of the toy model: In the low-energy limit the toy model predicts the mass ratios $m_{W_1} = m_{W_2} = m_{W_3}/2$. Furthermore, the toy model does not possess a $SO(4)$-symmetry in the intermediate range. These two facts are hardly compatible with observations. (For a survey on possible Lorentz-violating effects in QED see [14].) If we may neglect electromagnetic interactions we expect the weak gauge bosons to be degenerate in mass.

5 Improvement of the toy model

We may think about the manifold $S^1 \times S^3$ as a collection of three-dimensional slices S^3. Let us say that x_1, x_2 and x_3 are coordinates on S^3 and x_0 is the normal coordinate. For each slice the winding number of $S^3 \to SU(2)$ is fixed. For a non-trivial winding number we obtained a non-zero background field at $x_1 = x_2 = x_3 = 0$. Our toy model has the additional property that this background field stays constant as we go along the normal coordinate x_0. This is an unneccessary assumption. We may allow that the orientation of the background field changes as we pass along x_0 and replace B_μ by

$$B'_\mu = \Sigma(x_0)^{-1} B_\mu \Sigma(x_0) + \Sigma^{-1}(x_0)\partial_\mu \Sigma(x_0), \tag{39}$$

where $\Sigma(x_0)$ does depend on x_0, but not on x_1, x_2 or x_3. Let us assume that we are interested in proccesses with a scale k^2. This will probe a distance

$$\Delta x_0 = \frac{\lambda}{\sqrt{k^2}} \tag{40}$$

in the normal direction, where λ is some number between 0 and 1. Let us assume that Δx_0 is sufficiently large, such that the map

$$\begin{aligned}\Sigma \ : \ & [0,\Delta x_0] \rightarrow SU(2) \\ & x_0 \rightarrow \Sigma(x_0)\end{aligned} \tag{41}$$

sweeps out effectively all points in $SU(2)$-space. In this case it is reasonable to replace the x_0-dependent background field B'_μ by an average over all $SU(2)$-rotated configurations. To see this let us assume that Δx_0 is made out of n intervalls of length Δ, in which B_μ stays constant along the x_0 direction, and n transition intervalls of (negligible) length δ, in which the background field changes from one orientation to another. This can always be achieved by gauge transformations (connected to the identity). Symbolically we have

$$\begin{aligned}&\int\limits_{\Delta x_0} dx_0 \mathcal{L}(\Sigma(x_0)^{-1}B_\mu\Sigma(x_0)+\Sigma(x_0)^{-1}\partial_\mu\Sigma(x_0)) = \\ &= \ n\cdot\Delta\sum_i \mathcal{L}(\Sigma(x_0^i)^{-1}B_\mu\Sigma(x_0^i)) \\ &\quad + \int\limits_{n\times\delta} dx_0\mathcal{L}(\Sigma(x_0)^{-1}B_\mu\Sigma(x_0)+\Sigma(x_0)^{-1}\partial_\mu\Sigma(x_0)),\end{aligned} \tag{42}$$

where the sum is over all plateaux in which B'_μ is constant and x_0^i labels a point inside plateau i. Up to gauge transformations and reparametrization the second term on the r.h.s of eq. (42) is just $(n\cdot\delta/\Delta x_0)$ times the original integral. Therefore we established that we may replace the original integral over x_0 by an avergage over all $SU(2)$ configurations. Technically we do the averaging as follows: We replace B_μ by

$$B'_\mu \ = \ U^{-1}B_\mu U, \tag{43}$$

where

$$U \ = \ 1z_0 + i\sigma_1 z_1 + i\sigma_2 z_2 + i\sigma_3 z_3 \tag{44}$$

and

$$\begin{aligned} z_0 = \cos\alpha, \qquad & z_1 = \sin\alpha\cos\beta, \\ z_2 = \sin\alpha\sin\beta\cos\gamma, \qquad & z_3 = \sin\alpha\sin\beta\sin\gamma.\end{aligned} \tag{45}$$

In components we have

$$B^{a\prime}_\mu \ = \ \frac{1}{2}\mathrm{Tr}\,\left(U\sigma_a U^{-1}\sigma_b\right) B^b_\mu \tag{46}$$

We then integrate over $\alpha\in[0,\pi]$, $\beta\in[0,\pi]$ and $\gamma\in[0,2\pi]$ with the measure

$$d\,U \ = \ \sin^2\alpha\sin\beta\;d\alpha\;d\beta\;d\gamma. \tag{47}$$

We then obtain for the Lagrangian

$$\begin{aligned}\int d^4x\;(\mathcal{L}_{YM}+\mathcal{L}_{GF}) \ &= \ \frac{1}{2g^2}\int d^4x\;A^a_\mu\left(-\Box + 2\frac{m^2}{r^2}\right)A^a_\mu \\ &\quad +2if^{abc}A^a_\mu A^b_\nu\partial_\mu A^c_\nu - \frac{1}{2}f^{eac}f^{ebd}A^a_\mu A^b_\mu A^c_\nu A^d_\nu\end{aligned} \tag{48}$$

We observe that all terms linear in the background field have dropped out and that the $SU(2)$-gauge bosons have aquired a mass $\sqrt{2}m/r$. We also have observed that the effect of adding eq.(23) to the Lagrangian eq.(20) drops out after averaging. Therefore, in principle, we could weaken assumption 1 and/or eq.(21). Finally, we would like to remark that the effective Lagrangians eq.(28), eq.(36) and eq.(48) are not gauge invariant. Gauge invariance is broken by our choice of the gauge fixing term in eq.(27).

6 Conclusions

In this paper we investigated the effect of the conservation of topological winding numbers. We derived the effective theory in the background of a pure gauge field with non-trivial winding number. We showed that commutator terms may give rise to mass terms. Therefore a $U(1)$-theory stays massless, whereas a $SU(2)$-theory develops massive modes. In the most naive case we obtained an effective theory, which singles out a specific orientation in $SU(2)$-space. Averaging over all orientations we obtained a Lorentz-invariant effective theory where all $SU(2)$-gauge bosons aquire a mass $\sqrt{2}m/r$.

7 Acknowledgements

Many thanks to E. Laenen for organizing a working group at NIKHEF on effective theories, from which this work originated.

A The 't Hooft symbols

The 't Hooft symbols $\eta_{a\mu\nu}$ and $\bar{\eta}_{a\mu\nu}$ are defined as [6]

$$\begin{aligned} \eta_{a\mu\nu} = \bar{\eta}_{a\mu\nu} = \varepsilon_{a\mu\nu}, &\qquad a,\mu,\nu = 1,2,3 \\ \eta_{a\mu\nu} = -\eta_{a\nu\mu}, &\qquad \bar{\eta}_{a\mu\nu} = -\bar{\eta}_{a\nu\mu}, \\ \eta_{a\mu 0} = -\delta_{a\mu}, &\qquad \bar{\eta}_{a\mu 0} = \delta_{a\mu}. \end{aligned} \tag{49}$$

(Our notation differs slightly from 't Hooft, since we label the space-time coordinates by $0,1,2,3$, whereas 't Hooft uses $1,2,3,4$.) Our sign conventions for the antisymmetric tensors are: $\varepsilon_{123} = +1$, $\varepsilon_{0123} = +1$. The tensor $\eta_{a\mu\nu}$ is self-dual, whereas $\bar{\eta}_{a\mu\nu}$ is anti-self-dual:

$$\eta_{a\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\eta_{a\rho\sigma}, \qquad \bar{\eta}_{a\mu\nu} = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\bar{\eta}_{a\rho\sigma}. \tag{50}$$

We have the following relations:

$$\eta_{a\mu\nu}\eta_{b\mu\nu} = 4\delta_{ab}, \qquad \bar{\eta}_{a\mu\nu}\bar{\eta}_{b\mu\nu} = 4\delta_{ab}, \tag{51}$$

$$\eta_{a\mu\rho}\eta_{a\nu\rho} = 3\delta_{\mu\nu}, \qquad \bar{\eta}_{a\mu\rho}\bar{\eta}_{a\nu\rho} = 3\delta_{\mu\nu}, \tag{52}$$

$$\begin{aligned} \eta_{a\mu\nu}\eta_{a\rho\sigma} &= \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} + \varepsilon_{\mu\nu\rho\sigma}, \\ \bar{\eta}_{a\mu\nu}\bar{\eta}_{a\rho\sigma} &= \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho} - \varepsilon_{\mu\nu\rho\sigma}, \end{aligned} \tag{53}$$

$$\begin{aligned} \eta_{a\mu\rho}\eta_{b\nu\rho} &= \delta_{ab}\delta_{\mu\nu} + \varepsilon_{abc}\eta_{c\mu\nu}, \\ \bar{\eta}_{a\mu\rho}\bar{\eta}_{b\nu\rho} &= \delta_{ab}\delta_{\mu\nu} + \varepsilon_{abc}\bar{\eta}_{c\mu\nu}, \end{aligned} \tag{54}$$

$$\begin{aligned}
\varepsilon_{\mu\nu\rho\tau}\eta_{a\sigma\tau} &= \delta_{\mu\sigma}\eta_{a\nu\rho} + \delta_{\nu\sigma}\eta_{a\rho\mu} + \delta_{\rho\sigma}\eta_{a\mu\nu}, \\
\varepsilon_{\mu\nu\rho\tau}\bar{\eta}_{a\sigma\tau} &= -\delta_{\mu\sigma}\bar{\eta}_{a\nu\rho} - \delta_{\nu\sigma}\bar{\eta}_{a\rho\mu} - \delta_{\rho\sigma}\bar{\eta}_{a\mu\nu},
\end{aligned} \tag{55}$$

$$\begin{aligned}
\varepsilon_{abc}\eta_{b\mu\nu}\eta_{c\rho\sigma} &= \delta_{\mu\rho}\eta_{a\nu\sigma} - \delta_{\mu\sigma}\eta_{a\nu\rho} - \delta_{\nu\rho}\eta_{a\mu\sigma} + \delta_{\nu\sigma}\eta_{a\mu\rho}, \\
\varepsilon_{abc}\bar{\eta}_{b\mu\nu}\bar{\eta}_{c\rho\sigma} &= \delta_{\mu\rho}\bar{\eta}_{a\nu\sigma} - \delta_{\mu\sigma}\bar{\eta}_{a\nu\rho} - \delta_{\nu\rho}\bar{\eta}_{a\mu\sigma} + \delta_{\nu\sigma}\bar{\eta}_{a\mu\rho},
\end{aligned} \tag{56}$$

$$\begin{aligned}
\eta_{a\mu\nu}\bar{\eta}_{b\mu\nu} &= 0, \\
\eta_{a\mu\rho}\bar{\eta}_{b\nu\rho} - \eta_{a\nu\rho}\bar{\eta}_{b\mu\rho} &= 0.
\end{aligned} \tag{57}$$

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